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**MECHANIZATION EQUATIONS FOR A SCHULER-TUNED  
INERTIAL NAVIGATION SYSTEM VERTICALLY ALIGNED  
TO THE MASS-ATTRACTION GRAVITY VECTOR**

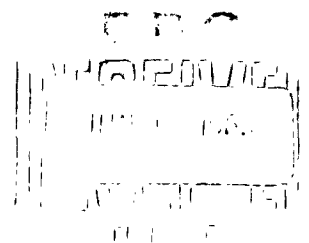
Philip C. Staas, Jr.

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GAM-87 Engineering Office  
Aeronautical Systems Division  
Air Force Systems Command  
Wright-Patterson Air Force Base, Ohio

Weapon System 138A



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## FOREWORD

This report was prepared by Philip C. Staas, Jr., of the Guidance Branch, GAM-87 Engineering Office, Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio. Interest to develop the material in this report was stimulated through participation in the numerous performance analyses conducted on the GAM-87 Guidance System by the GAM-87 Engineering Office, Guidance Branch personnel. The work was documented under Weapon System 138A.

The studies presented began in September 1962 and were completed in December 1962.

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### ABSTRACT

A navigation scheme is investigated that uses a Schuler-tuned inertial platform that is maintained orthogonal to the mass-attraction gravity vector. The expressions for platform acceleration and velocity are developed, and a mechanization to achieve navigation is determined. Results indicate that an inertial platform that is not torqued in the vertical axis can be readily mechanized to seek a mass-attraction vertical and to accomplish navigation on earth.

### PUBLICATION REVIEW

Publication of this technical documentary report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

## TABLE OF CONTENTS

|  | Page |
|--|------|
| Introduction . . . . .   | 1    |
| Derivation of Mass Acceleration in a Rotating Coordinate System . . . . .                      | 1    |
| Orientation of a Coordinate System Along the Mass-Attraction Vector . .                        | 3    |
| Mechanization of Acceleration Expressions to Achieve Inertial Velocity<br>of Point P . . . . . | 6    |
| Determination of Platform Position in Coordinate System of the Earth . .                       | 11   |
| Conclusions . . . . .  | 13   |
| Bibliography . . . . .   | 13   |

## INTRODUCTION

To navigate in the geographic coordinate system of the earth, we desire to use accelerometers that are mounted on an inertial platform. The total acceleration in inertial space as experienced by the platform that is positioned at a point P, which is a distance R from the center of the earth, will be derived. A particular orientation of the platform is chosen so that its vertical axis is coincident with the vector  $\bar{R}$  from the center of the earth. In the development, the geographic center of the earth and the center of mass attraction will coincide so that the platform will be aligned to the mass-attraction gravity vector.

The position of the platform on the earth will be expressed in terms of the geographic longitude angle and the mass-attraction latitude angle. Note that in contrast the geographic latitude is measured from a plumb bob vertical. The mass suspended by a plumb line is attracted by the gravitation of the earth and is influenced by the rotation of the earth to cause the plumb line to point along the vector sum of the mass-attraction vector and the centripetal acceleration vector. To account for a discrepancy in latitude position resulting from a determination of position using mass-attraction vertical and a determination of position using a plumb bob vertical, we will alter the geographic inputs and outputs to the platform for the contribution of centripetal acceleration.

To mechanize the expressions derived for acceleration and velocity of the platform, we will use the Schuler principle of developing an angular rate of the platform that is equal to the inertial velocity divided by the radius of the earth to maintain a mass-attraction vertical and to navigate over the earth.

DERIVATION OF MASS ACCELERATION  
IN A ROTATING COORDINATE SYSTEM

A right-hand coordinate system is selected with its origin at the center of the earth, its  $\bar{K}$  axis through the North Pole of the earth, and its  $\bar{I}$  and  $\bar{J}$  axes in the plane of the earth's equator with  $\bar{I}$  passing through the Greenwich meridian. In this development, the radius of the earth is considered constant for all angles above or below the equatorial plane. See Figure 1.

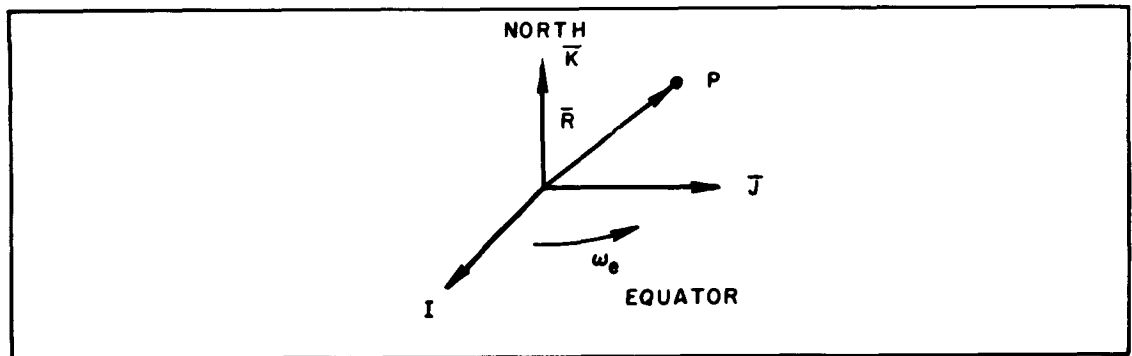


Figure 1. Coordinate System With Origin at Center of Earth



We desire to determine the acceleration in inertial space that is experienced at point P while P is free to move in the  $\bar{I}$ ,  $\bar{J}$ , and  $\bar{K}$  coordinate system and while the  $\bar{I}$ ,  $\bar{J}$ , and  $\bar{K}$  axes rotate in inertial space about  $\bar{K}$  with an angular rate  $\omega_e$ . In the development, the following terms will be used:

$$\bar{\omega}_e = \omega_e \bar{K}$$

$$\bar{R} = \bar{I}R_x + \bar{J}R_y + \bar{K}R_z$$

$$\dot{\bar{R}} = \text{inertial velocity of P}$$

$$\bar{V}_p = \text{velocity of P relative to } \bar{I}, \bar{J}, \text{ and } \bar{K} \text{ axes}$$

$$\ddot{\bar{R}} = \text{inertial acceleration of P}$$

$$\bar{a}_p = \text{acceleration of P relative to } \bar{I}, \bar{J}, \text{ and } \bar{K} \text{ axes}$$

$$\bar{A}_I = \text{acceleration of P in inertial space including gravity of mass attraction}$$

$$\bar{g}_m = \text{gravity vector of mass attraction of earth assumed directed toward the center of the earth.}$$

$\bar{R}$  will be differentiated once to form  $\dot{\bar{R}}$ , the inertial velocity of P, and once again to form  $\ddot{\bar{R}}$ , the inertial acceleration of P. Therefore,

$$\bar{R} = \bar{I}R_x + \bar{J}R_y + \bar{K}R_z$$

$$\dot{\bar{R}} = \dot{\bar{I}}R_x + \dot{\bar{J}}R_y + \dot{\bar{K}}R_z + \bar{I}\dot{R}_x + \bar{J}\dot{R}_y + \bar{K}\dot{R}_z$$

$$\dot{\bar{R}} = \bar{\omega}_e \times \bar{R} + \bar{V}_p$$

$$\ddot{\bar{R}} = \dot{\bar{\omega}}_e \times \bar{R} + \bar{\omega}_e \times \dot{\bar{R}} + \dot{\bar{V}}_p$$

$$\dot{\bar{V}}_p = \dot{\bar{I}}\dot{R}_x + \dot{\bar{J}}\dot{R}_y + \dot{\bar{K}}\dot{R}_z + \bar{I}\ddot{R}_x + \bar{J}\ddot{R}_y + \bar{K}\ddot{R}_z$$

$$\dot{\bar{V}}_p = \bar{\omega}_e \times \bar{V}_p + \bar{a}_p$$

$$\dot{\bar{\omega}}_e = 0, \bar{\omega}_e \text{ considered constant}$$

$$\ddot{\bar{R}} = \bar{\omega}_e \times \dot{\bar{R}} + \dot{\bar{V}}_p$$

$$\ddot{\bar{R}} = \bar{\omega}_e \times (\bar{\omega}_e \times \bar{R} + \bar{V}_p) + \bar{\omega}_e \times \bar{V}_p + \bar{a}_p$$

$$\ddot{\bar{R}} = \bar{\omega}_e \times \bar{\omega}_e \times \bar{R} + 2\bar{\omega}_e \times \bar{V}_p + \bar{a}_p$$

$$\bar{A}_I = \ddot{\bar{R}} + \bar{g}_m$$

$$\bar{A}_I = \bar{\omega}_e \times \bar{\omega}_e \times \bar{R} + 2\bar{\omega}_e \times \bar{V}_p + \bar{a}_p + \bar{g}_m$$

The preceding expression for  $\bar{A}_I$  is the acceleration of P in inertial space, which differs from  $\ddot{\bar{R}}$  by the gravity term  $\bar{g}_m$ . A plumb bob on the earth will be under the influence of  $\bar{A}_I$  to establish a vertical direction, which is the vector sum of  $\ddot{\bar{R}}$  +  $\bar{g}_m$ . Considerations of the acceleration of center of the earth relative to the center of inertial space are neglected in  $\bar{A}_I$  as insignificant.

$\bar{A}_I - \bar{g}_m$  in terms of  $\bar{I}$ ,  $\bar{J}$ , and  $\bar{K}$  is as follows:

$$\begin{aligned}\bar{\omega}_e \times \bar{R} &= \bar{I} (-\omega_e R_y) + \bar{J} (\omega_e R_x) \\ \bar{\omega}_e \times \bar{\omega}_e \times \bar{R} &= \bar{I} (-\omega_e^2 R_x) + \bar{J} (-\omega_e^2 R_y) \\ 2\bar{\omega}_e \times \bar{V}_p &= 2\bar{I} (-\omega_e \dot{R}_y) + 2\bar{J} (\omega_e \dot{R}_x) \\ \bar{A}_I &= \bar{I} [(-\omega_e^2 R_x) + 2(-\omega_e \dot{R}_y) + \ddot{R}_x] \\ &\quad + \bar{J} [(-\omega_e^2 R_y) + 2(\omega_e \dot{R}_x) + \ddot{R}_y] \\ &\quad + \bar{K} [\ddot{R}_z] \\ &\quad + \bar{g}_m.\end{aligned}$$

#### ORIENTATION OF A COORDINATE SYSTEM ALONG THE MASS-ATTRACTION VECTOR

We desire to orient a coordinate system  $\bar{I}$ ,  $\bar{J}$ , and  $\bar{K}$  at P so that  $\bar{K}$  is along  $\bar{R}$ ,  $\bar{I}$  is directed toward  $\bar{K}$  (North), and  $\bar{J}$  then is directed eastward. See Figure 2.

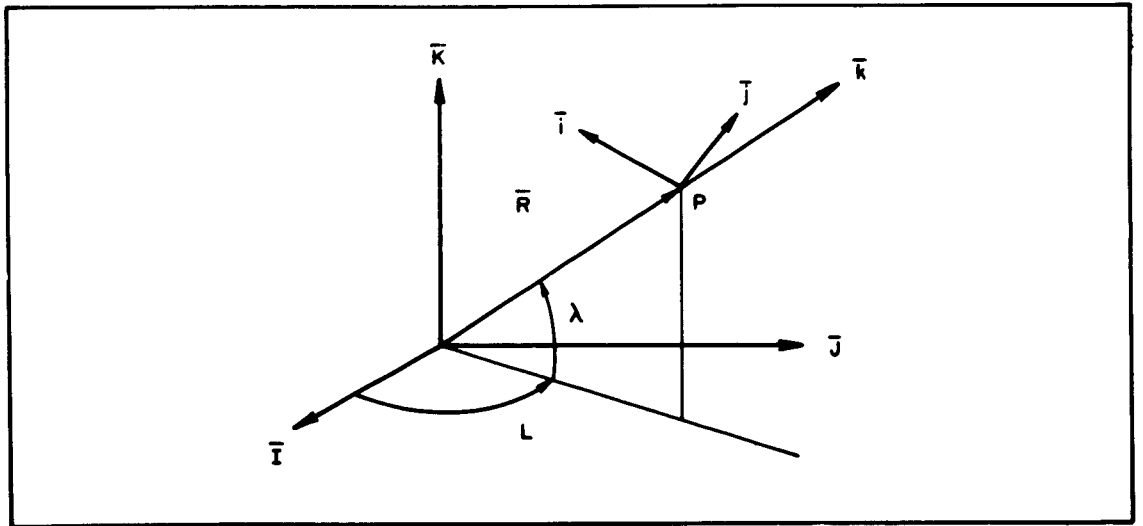


Figure 2. Coordinate System With One Axis Directed Toward Center of Earth

In this development, the following terms will be used:

$L$  = angle eastward from Greenwich, longitude

$\lambda$  = angle from the equatorial plane, mass-attraction latitude

$$\bar{k} = \bar{i} \cos \lambda \cos L + \bar{j} \cos \lambda \sin L + \bar{k} \sin \lambda$$

$$\bar{j} = \bar{i} (-\sin L) + \bar{j} (\cos L)$$

$$\bar{i} = \bar{i} (-\sin \lambda \cos L) + \bar{j} (-\sin \lambda \sin L) + \bar{k} \cos \lambda.$$

The intent of positioning  $\bar{k}$  to be coincident with  $\bar{R}$  is to provide a coordinate system with one axis directed toward the center of the earth. When the geographical center of the earth is assumed to be located at the center of mass attraction,  $\bar{i}$ ,  $\bar{j}$ , and  $\bar{k}$  will then be aligned along the mass-attraction vector.

The acceleration of point P is now to be expressed in terms of  $\bar{A}_I$  along  $\bar{i}$ ,  $\bar{j}$ , and  $\bar{k}$  as follows:

$$\begin{aligned} \bar{A}_I = & \bar{i} \left[ (\omega_e^2 R_x \sin \lambda \cos L + \omega_e^2 R_y \sin \lambda \sin L) \right. \\ & + (2\omega_e \dot{R}_y \sin \lambda \cos L - 2\omega_e \dot{R}_x \sin \lambda \sin L) \\ & + (\ddot{R}_x \sin \lambda \cos L + \ddot{R}_y \sin \lambda \sin L - \ddot{R}_z \cos \lambda) \left. \right] \\ & + \bar{j} \left[ (\omega_e^2 R_x \sin L - \omega_e^2 R_y \cos L) \right. \\ & + (2\omega_e \dot{R}_y \sin L + 2\omega_e \dot{R}_x \cos L) \\ & + (\ddot{R}_x \sin L - \ddot{R}_y \cos L) \left. \right] \\ & + \bar{k} \left[ -(\omega_e^2 R_x \cos \lambda \cos L + \omega_e^2 R_y \cos \lambda \sin L) \right. \\ & - (2\omega_e \dot{R}_y \cos \lambda \cos L - 2\omega_e \dot{R}_x \cos \lambda \sin L) \\ & + (\ddot{R}_x \cos \lambda \cos L + \ddot{R}_y \cos \lambda \sin L + \ddot{R}_z \sin \lambda) \\ & \left. + g_m \right]. \end{aligned}$$

To express  $\bar{A}_I$  in terms of  $R$ ,  $\dot{R}$ , and  $\ddot{R}$ , we must derive the following equalities:

$$R_x = R \cos \lambda \cos L$$

$$R_y = R \cos \lambda \sin L$$

$$R_z = R \sin \lambda$$

$$R_y \cos L = R_x \sin L$$

$$R_y \cos L - R_x \sin L = 0$$

$$R_x \cos L + R_y \sin L = R \cos \lambda$$

$$\dot{R}_x \cos L + \dot{R}_y \sin L = \dot{R} \cos \lambda - R \dot{\lambda} \sin \lambda$$

$$\dot{R}_y \cos L - \dot{R}_x \sin L = R \dot{L} \cos \lambda$$

$$\begin{aligned} (\ddot{R}_x \cos L + \ddot{R}_y \sin L) \sin \lambda &= -R \sin \lambda \cos \lambda \left[ (\dot{L})^2 + (\dot{\lambda})^2 \right] \\ &\quad - \sin^2 \lambda (2\dot{R}\dot{\lambda} + R\ddot{\lambda}) + \ddot{R} \cos \lambda \sin \lambda \end{aligned}$$

$$\begin{aligned} (\ddot{R}_x \cos L + \ddot{R}_y \sin L) \cos \lambda &= -R \cos^2 \lambda \left[ (\dot{L})^2 + (\dot{\lambda})^2 \right] \\ &\quad - \cos \lambda \sin \lambda (2\dot{R}\dot{\lambda} + R\ddot{\lambda}) + \ddot{R} \cos^2 \lambda \end{aligned}$$

$$\ddot{R}_y \cos L - \ddot{R}_x \sin L = R \ddot{L} \cos \lambda - 2R \dot{L} \dot{\lambda} \sin \lambda + 2R \dot{L} \cos \lambda$$

$$\dot{R}_z = \dot{R} \sin \lambda + R \dot{\lambda} \cos \lambda$$

$$\ddot{R}_z = \ddot{R} \sin \lambda + 2\dot{R}\dot{\lambda} \cos \lambda + R\ddot{\lambda} \cos \lambda - R(\dot{\lambda})^2 \sin \lambda$$

Then substituting the equalities gives

$$\begin{aligned} \bar{A}_I &= \\ &\bar{i} \left[ R \sin \lambda \cos \lambda (\omega_e + \dot{L})^2 + 2\dot{R}\dot{\lambda} + R\ddot{\lambda} \right] \\ &+ \bar{j} \left[ 2(\omega_e + \dot{L}) \dot{R} \cos \lambda - 2(\omega_e + \dot{L}) R \dot{\lambda} \sin \lambda + R \ddot{L} \cos \lambda \right] \\ &+ \bar{k} \left[ -(\omega_e + \dot{L})^2 R \cos^2 \lambda - R(\dot{\lambda})^2 + \ddot{R} + g_m \right] \end{aligned}$$

# MECHANIZATION OF ACCELERATION EXPRESSIONS TO ACHIEVE INERTIAL VELOCITY OF POINT P

We desire to integrate the acceleration expressions derived so that the velocity of P in inertial space is expressed in a coordinate system  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$  that is related to  $\bar{i}$ ,  $\bar{j}$ , and  $\bar{k}$  as follows:

$$\bar{r}_1 = \bar{k}$$

$$\bar{r}_2 = \bar{i} \sin \Psi_p + \bar{j} \cos \Psi_p$$

$$\bar{r}_3 = \bar{i} \cos \Psi_p - \bar{j} \sin \Psi_p$$

Figure 3 shows the desired coordinate system.

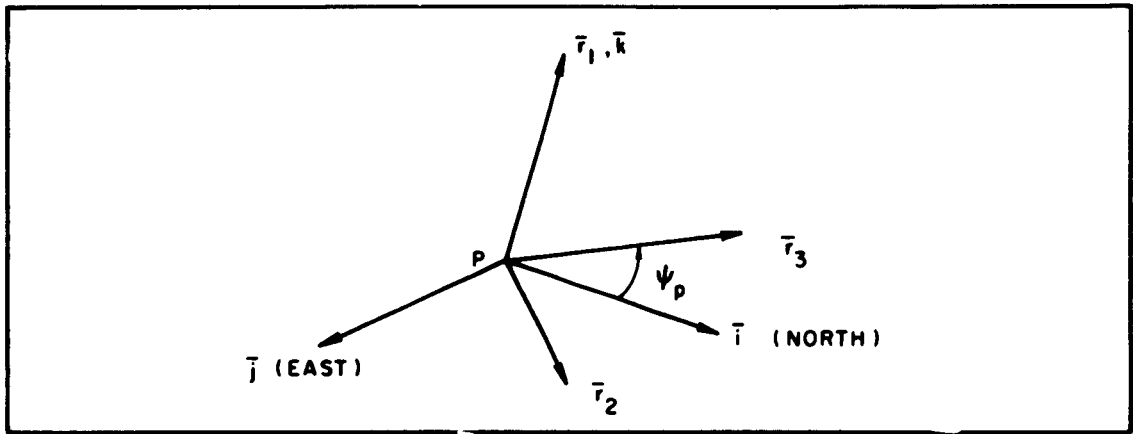


Figure 3. Coordinate System  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$

Angle  $\Psi_p$ , measured counterclockwise from  $\bar{i}$  (North), at this step in the development will not be restricted to a magnitude or rate ( $\dot{\Psi}_p$ ). Eventually  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$  will be restricted so that there will be no rotation of the coordinate system about  $\bar{r}_1$  in inertial space.

$\bar{A}_1$  is now expressed in the  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$  directions:

$$\begin{aligned} \bar{A}_1 = & \bar{r}_1 \left[ -(\omega_e + \dot{L})^2 R \cos^2 \lambda - R(\dot{\lambda})^2 + \ddot{R} + g_m \right] \\ & + \bar{r}_2 \left[ (\omega_e + \dot{L})^2 R \sin \lambda \cos \lambda \sin \Psi_p + 2\dot{R}\dot{\lambda} \sin \Psi_p \right. \\ & + 2(\omega_e + \dot{L})\dot{R} \cos \lambda \cos \Psi_p + R\ddot{\lambda} \sin \Psi_p \\ & \left. - 2(\omega_e + \dot{L})R\dot{\lambda} \sin \lambda \cos \Psi_p + R\ddot{L} \cos \lambda \cos \Psi_p \right] \end{aligned}$$

$$\begin{aligned}
& + \bar{r}_3 \left[ (\omega_e + \dot{L})^2 R \sin \lambda \cos \lambda \cos \Psi_p + 2\dot{R}\dot{\lambda} \cos \Psi_p \right. \\
& - 2(\omega_e + \dot{L}) \dot{R} \cos \lambda \sin \Psi_p + R\ddot{\lambda} \cos \Psi_p \\
& \left. 2(\omega_e + \dot{L}) R \dot{\lambda} \sin \lambda \sin \Psi_p - R\ddot{L} \cos \lambda \sin \Psi_p \right]
\end{aligned}$$

Upon integration of  $\bar{A}_1$ , the inertial velocity of P should be equal to  $\dot{\bar{R}}$ , which was previously derived. Therefore,

$$\dot{\bar{R}} = \bar{\omega}_e \times \bar{R} + \bar{V}_p$$

$$\bar{\omega}_e \times \bar{R} = \bar{I}(-\omega_e R_y) + \bar{J}(\omega_e R_x)$$

$$\bar{\omega}_e \times \bar{R} = \bar{j} \omega_e R \cos \lambda$$

$$\bar{V}_p = \bar{I} \dot{R}_x + \bar{J} \dot{R}_y + \bar{K} \dot{R}_z$$

$$\bar{V}_p = \bar{I} R \dot{\lambda} + \bar{j} R \dot{L} \cos \lambda + \bar{k} \dot{R}$$

$$\dot{\bar{R}} = \bar{I} R \dot{\lambda} + \bar{j} (\omega_e + \dot{L}) R \cos \lambda + \bar{k} \dot{R}$$

$$\begin{aligned}
\dot{\bar{R}} &= \bar{r}_3 \left[ R \dot{\lambda} \cos \Psi_p - (\omega_e + \dot{L}) R \cos \lambda \sin \Psi_p \right] \\
&+ \bar{r}_2 \left[ R \dot{\lambda} \sin \Psi_p + (\omega_e + \dot{L}) R \cos \lambda \cos \Psi_p \right] \\
&+ \bar{r}_1 \left[ \dot{R} \right]
\end{aligned}$$

$\dot{\bar{R}}$  is to be the output of three isolated integrators following three accelerometers aligned along  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$ , respectively, which sense  $\bar{A}_1$ . Differentiating  $\dot{\bar{R}}$  to achieve  $\ddot{\bar{R}}$  will give the expression for the accelerations required at the input to the integrators. Therefore,

$$\ddot{\bar{R}} =$$

$$\begin{aligned}
& \bar{r}_3 \left[ R \dot{\lambda} \cos \Psi_p - (\omega_e + \dot{L}) R \cos \lambda \sin \Psi_p \right] \\
& + \bar{r}_3 \left[ \dot{R} \dot{\lambda} \cos \Psi_p + R \ddot{\lambda} \cos \Psi_p - (\omega_e + \dot{L}) \dot{R} \cos \lambda \sin \Psi_p \right. \\
& + (\omega_e + \dot{L}) R \dot{\lambda} \sin \lambda \sin \Psi_p - \ddot{L} R \cos \lambda \sin \Psi_p \\
& \left. - R \dot{\lambda} \dot{\Psi}_p \sin \Psi_p - (\omega_e + \dot{L}) R \dot{\Psi}_p \cos \lambda \cos \Psi_p \right]
\end{aligned}$$

$$\begin{aligned}
& + \dot{\bar{r}}_2 \left[ R \dot{\lambda} \sin \Psi_p + (\omega_e + \dot{L}) R \cos \lambda \cos \Psi_p \right] \\
& + \bar{r}_2 \left[ \ddot{R} \dot{\lambda} \sin \Psi_p + R \ddot{\lambda} \sin \Psi_p + (\omega_e + \dot{L}) R \cos \lambda \cos \Psi_p \right. \\
& - (\omega_e + \dot{L}) R \dot{\lambda} \sin \lambda \cos \Psi_p + \ddot{L} R \cos \lambda \cos \Psi_p \\
& \left. R \dot{\lambda} \dot{\Psi}_p \cos \Psi_p - (\omega_e + \dot{L}) R \dot{\Psi}_p \cos \lambda \sin \Psi_p \right] \\
& + \dot{\bar{r}}_1 \left[ \dot{R} \right] + \bar{r}_1 \left[ \ddot{R} \right]
\end{aligned}$$

$\dot{\Psi}_p$  in the expression for  $\ddot{\bar{R}}$  is evaluated as follows:

$$\begin{aligned}
\bar{\omega}_e &= \omega_e \bar{K} \\
\bar{\omega}_e &= \bar{i} \omega_e \cos \lambda + \bar{k} \omega_e \sin \lambda
\end{aligned}$$

From the preceding expression, a rotation  $\omega_e \sin \lambda$  occurs about  $\bar{k}$  in inertial space. The restriction that the  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$  coordinate system cannot rotate in inertial space about  $\bar{r}_1$  will now be incorporated in the development. When  $\bar{r}_1$  is coincident with  $\bar{k}$  and  $\Psi_p$  is the angle between  $\bar{i}$  and  $\bar{r}_3$ ,

$$\dot{\Psi}_p = -(\omega_e + \dot{L}) \sin \lambda.$$

The rate of  $\dot{\Psi}_p$  is negative because positive rotation was selected counterclockwise for  $\omega_e$ . To accomplish no rotation in inertial space about  $\bar{r}_1$ , we will not torque the inertial platform in this axis.

The  $\dot{\bar{r}}$  terms in  $\ddot{\bar{R}}$  are evaluated as follows:

$$\begin{aligned}
\bar{\omega}_p \times \dot{\bar{R}} &= \dot{\bar{r}}_3 \left[ R \dot{\lambda} \cos \Psi_p - (\omega_e + \dot{L}) R \cos \lambda \sin \Psi_p \right] \\
&+ \dot{\bar{r}}_2 \left[ R \dot{\lambda} \sin \Psi_p + (\omega_e + \dot{L}) R \cos \lambda \cos \Psi_p \right] \\
&+ \dot{\bar{r}}_1 \left[ \dot{R} \right]
\end{aligned}$$

where  $\bar{\omega}_p$  is the rotation of  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$  in inertial space. To accomplish navigation in the coordinate system of the earth, we will follow the Schuler principle for generation of  $\bar{\omega}_p$ . In a Schuler mechanization, the integrated acceleration, which is velocity, for each of the two horizontal axes ( $\bar{r}_2$  and  $\bar{r}_3$ ) is divided by  $R$ , the radius of the earth, which gives an angular rate,  $\omega$ , for torquing the inertial platform to maintain the aforementioned axes in the horizontal plane. Following this procedure, we obtain

$$\omega_3 \bar{r}_3 = \frac{V_2}{R} \bar{r}_2$$

and

$$\omega_2 \bar{r}_2 = -\frac{V_3}{R} \bar{r}_3$$

Note that  $\omega_2$  is negative to maintain  $\bar{r}_1$  horizontal.

The  $V_1$ ,  $V_2$ , and  $V_3$  inertial velocities are found from  $\dot{\bar{R}}$ :

$$\begin{aligned} V_2 &= R \left[ \dot{\lambda} \sin \Psi_p + (\omega_e + \dot{L}) \cos \lambda \cos \Psi_p \right] \\ V_3 &= R \left[ \dot{\lambda} \cos \Psi_p - (\omega_e + \dot{L}) \cos \lambda \sin \Psi_p \right] \\ V_1 &= \left[ \dot{R} \right] \end{aligned}$$

Note that  $\omega_1 \bar{r}_1$  is zero, therefore

$$\begin{aligned} \bar{\omega}_p &= \omega_3 \bar{r}_3 + \omega_2 \bar{r}_2 \\ \dot{\bar{R}} &= V_3 \bar{r}_3 + V_2 \bar{r}_2 + V_1 \bar{r}_1 \\ \bar{\omega}_p \times \dot{\bar{R}} &= \left[ \omega_2 V_3 - \omega_3 V_2 \right] \bar{r}_1 + \left[ \omega_3 V_1 \right] \bar{r}_2 - \left[ \omega_2 V_1 \right] \bar{r}_3 \\ \bar{\omega}_p \times \dot{\bar{R}} &= \bar{r}_1 \left[ -R(\dot{\lambda})^2 - (\omega_e + \dot{L})^2 R \cos^2 \lambda \right] \\ &\quad + \bar{r}_2 \left[ (\dot{R}) \right] \left[ \dot{\lambda} \sin \Psi_p + (\omega_e + \dot{L}) \cos \lambda \cos \Psi_p \right] \\ &\quad + \bar{r}_3 \left[ (\dot{R}) \right] \left[ \dot{\lambda} \cos \Psi_p - (\omega_e + \dot{L}) \cos \lambda \sin \Psi_p \right] \end{aligned}$$

Substituting expressions developed for  $\dot{\Psi}_p$  and  $\dot{\bar{r}}$  terms gives

$$\begin{aligned} \ddot{\bar{R}} &= \bar{r}_3 \left[ \dot{R} \dot{\lambda} \cos \Psi_p + R \ddot{\lambda} \cos \Psi_p - (\omega_e + \dot{L}) \dot{R} \cos \lambda \sin \Psi_p \right. \\ &\quad + 2(\omega_e + \dot{L}) R \dot{\lambda} \sin \lambda \sin \Psi_p - \ddot{L} R \cos \lambda \sin \Psi_p \\ &\quad \left. + (\omega_e + \dot{L})^2 R \sin \lambda \cos \lambda \cos \Psi_p \right] \\ &\quad + \bar{r}_2 \left[ \dot{R} \dot{\lambda} \sin \Psi_p + R \ddot{\lambda} \sin \Psi_p + (\omega_e + \dot{L}) \dot{R} \cos \lambda \cos \Psi_p \right. \\ &\quad - 2(\omega_e + \dot{L}) R \dot{\lambda} \sin \lambda \cos \Psi_p + \ddot{L} R \cos \lambda \cos \Psi_p \\ &\quad \left. + (\omega_e + \dot{L})^2 R \sin \lambda \cos \lambda \sin \Psi_p \right] \\ &\quad + \bar{r}_1 \left[ \ddot{R} \right] \\ &\quad + \bar{\omega}_p \times \dot{\bar{R}} \end{aligned}$$



If the value of  $\bar{\omega}_p \times \dot{\bar{R}}$  were included in the  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$  directions above for  $\ddot{\bar{R}}$ , we would see that  $\ddot{\bar{R}} = \bar{A}_I - \bar{g}_m$ .

At this point in the development, we must examine  $\bar{A}_I$  and the accelerometers in the  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$  coordinate system. For accomplishment of the Schuler mechanization, the platform is rotated in inertial space by  $\bar{\omega}_p$ . Torque is applied to the platform at a rate  $\bar{\omega}_p$  in the two horizontal axes to oppose the inertial position sought by the platform so that an earth orientation can be maintained. Because the platform and the accelerometers have an inertial velocity  $\dot{\bar{R}}$ , a rotation of the platform coordinate system in inertial space results in an apparent acceleration of the platform equal to  $\bar{\omega}_p \times \dot{\bar{R}}$  as seen in the coordinate system of the earth. The contribution of  $\bar{\omega}_p \times \dot{\bar{R}}$  to  $\ddot{\bar{R}}$  is determined from the total integral of  $\bar{A}_I - \bar{g}_m$ :

$$\begin{aligned}\dot{\bar{R}} &= \int (\bar{A}_I - \bar{g}_m) dt \\ &= \left[ \int (A_1 - g_m) dt \right] \bar{r}_1 + \left[ \int A_2 dt \right] \bar{r}_2 \\ &\quad + \left[ \int A_3 dt \right] \bar{r}_3 + \int (\bar{\omega}_p \times \dot{\bar{R}}) dt\end{aligned}$$

Note that the above expression for  $\bar{A}_I$  was verified by differentiating  $\dot{\bar{R}}$ .

The preceding expression clearly shows the components of the total  $\bar{A}_I$  that will be sensed by accelerometers aligned along  $\bar{r}_1$ ,  $\bar{r}_2$ , and  $\bar{r}_3$ , namely  $A_1$ ,  $A_2$ , and  $A_3$ , respectively. To accomplish the total integration of  $\bar{A}_I$ , we must supply the  $\bar{\omega}_p \times \dot{\bar{R}}$  term at the input to the three integrators receiving  $A_1$ ,  $A_2$ , and  $A_3$ . In the mechanization of the desired inertial navigator, therefore, the velocity of point P can be achieved from the accelerometers by adding the following quantities:

To accelerometer output  $A_1$  must be added

$$\left[ \dot{(\bar{R})} \right] \left[ \dot{\lambda} \cos \Psi_p - (\omega_e + \dot{L}) \cos \lambda \sin \Psi_p \right]$$

which is equal to  $-\omega_2 V_1$ .

To accelerometer output  $A_2$  must be added

$$\left[ \dot{(\bar{R})} \right] \left[ \dot{\lambda} \sin \Psi_p + (\omega_e + \dot{L}) \cos \lambda \cos \Psi_p \right]$$

which is equal to  $\omega_3 V_1$ .

To accelerometer output  $A_1$  must be added

$$\left[ -R(\dot{\lambda})^2 - (\omega_e + \dot{L})^2 R \cos^2 \lambda - g_m \right]$$

which is equal to  $\omega_2 V_3 - \omega_3 V_2 - g_m$ .

In the closed-loop operation of the inertial platform, Figure 4, the  $\bar{\omega}_p$  and  $\dot{R}$  terms that must supply  $\bar{\omega}_p \times \dot{R}$  are readily available as outputs of the platform.

#### DETERMINATION OF PLATFORM POSITION IN THE COORDINATE SYSTEM OF THE EARTH

Navigation in the coordinate system of the earth is accomplished by using the torquing rates,  $\omega_2$  and  $\omega_3$ , that are generated in the mechanization of the Schuler-tuned inertial navigation system as follows:

Mass attraction latitude results from an integration of  $\dot{\lambda}$ , where,

$$\dot{\lambda} = \omega_2 \cos \psi_p + \omega_3 \sin \psi_p,$$

and longitude results from an integration of  $\dot{L}$ , where,

$$\dot{L} = \frac{\omega_3 \cos \psi_p - \omega_2 \sin \psi_p - \omega_e}{\cos \lambda}.$$

In these expressions,  $\omega_e$ , rotation of the earth, is a constant, and  $\psi_p$  is determined from an integration of  $\dot{\psi}_p$ , where,

$$\dot{\psi}_p = -(\omega_e + \dot{L}) \sin \lambda.$$

Mass attraction latitude is transformed to geographic latitude as follows:

$$\bar{g} = \bar{a}_c + \bar{g}_m$$

where

$\bar{g}$  = plumb bob vertical

$\bar{a}_c$  = earth's centripetal acceleration component that is perpendicular to  $\bar{g}_m$

$\bar{g}_m$  = mass-attraction vertical

$\alpha$  = angle between  $\bar{g}$  and  $\bar{g}_m$ .

Note that  $\alpha < 1^\circ$ , then

$$\alpha = \frac{a_c}{g_m} = \frac{(\omega_e)^2 R \sin \lambda \cos \lambda}{g_m} \quad \text{IN RADIANS.}$$

Angle  $\alpha$  is added to the mass-attraction vertical angle for latitude to obtain geographic latitude.

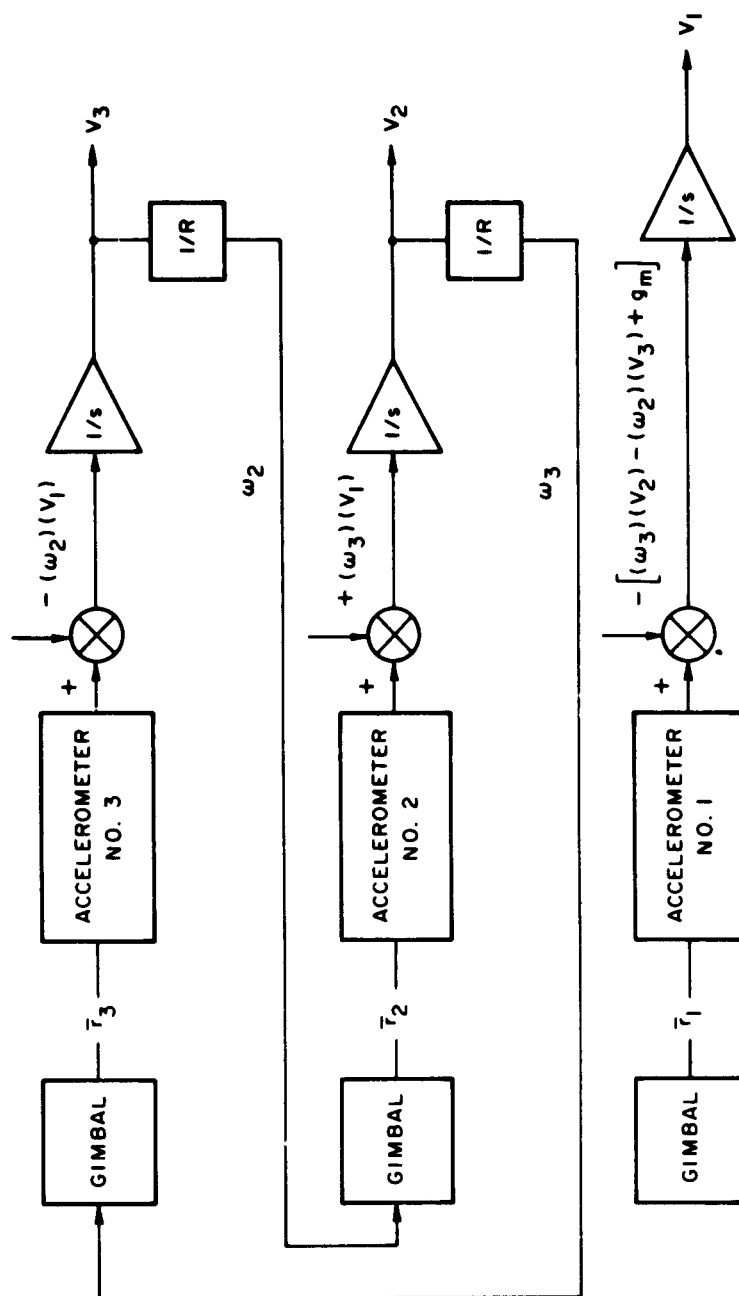


Figure 4. Mechanization of Velocity Expressions

## CONCLUSIONS

The mechanization of inertially measured accelerations to derive inertial velocity and to accomplish navigation on earth is readily achieved from three accelerometers that are orthogonally mounted in a platform coordinated system, which is vertically aligned to the mass-attraction vector of the earth and which is not torqued about the vertical axis.

To obtain platform velocity relative to the earth, we must subtract the velocity of the earth,  $\bar{\omega}_e \times \bar{R}$ , from the inertial velocity output of the platform. Because latitude position that is determined by the inertial navigator does not include centripetal acceleration, an angular displacement must be added to the latitude output of the platform to obtain geographic latitude.

The radius of the earth was assumed constant in the development. Varying  $R$  as a function of latitude can account for the spheroidal shape of the earth. To include the altitude of the platform above the surface of the earth in the total distance of the platform from the center of the earth, we can add an integration of platform velocity in the vertical,  $\bar{r}_1$ , direction to  $R$ .

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( over )

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